MATHEMATICS  March, 2008

Answer all the ten questions :  

1. Find the least positive integer \( x \) satisfying \( 2x + 5 = x + 4 \pmod{5} \).

2. If \( A = \begin{bmatrix} 5-x & 2y-8 \\ 0 & 3 \end{bmatrix} \) is scalar matrix, find \( x \) and \( y \).

3. If \( a \ast b = \frac{3ab}{7} \), then prove that \( \ast \) is associative.

4. Define co-planar vectors.

5. Write the condition for the circle \( x^2 + y^2 + 2gx + 2fy + c = 0 \) touches both axes.

6. Find the co-ordinates of the end points of length of the latus rectum of the parabola \( y^2 = 12x \).

7. Find the value of \( \tan \left( \tan^{-1} 3 \right) + \sec^{-1} \left\{ \sec (-2) \right\} \).

8. Write the multiplicative inverse of \( i \).

9. Define the differential coefficient of a continuous function \( y = f(x) \) w.r.t. \( x \).

10. Evaluate \( \int \frac{1 - \cos x}{\sin^2 x} \, dx \).

PART – B

Answer any ten questions :  

11. The relation ‘Congruence modulo \( m \)’ is an equivalence relation on \( \mathbb{Z} \) or prove that \( a \equiv b \pmod{m} \) is an equivalence relation on \( \mathbb{Z} \).
13. If in a group \((G, \ast)\) \(\forall a \in G, \quad a^{-1} = a\), then prove that \((G, \ast)\) is an Abelian group.

14. If the vectors \(\lambda \hat{i} + 2\hat{j} - \hat{k}\) and \(\hat{i} - 3\hat{j} + 2\hat{k}\) are orthogonal, find \(\lambda\).

15. Find the area of the circle whose parametric equations are

\[ x = 3 + 2\cos \theta \quad \text{and} \quad y = 1 + 2\sin \theta. \]

16. Find the equation of the hyperbola in the form \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.\) Given that transverse axis = 10, and eccentricity \((e) = 2.\)

17. Find \(x\) if \(\tan^{-1} = \sin^{-1}\frac{1}{2} + \cos^{-1}\frac{\sqrt{3} + 1}{2\sqrt{2}}.\)

18. Prove that \(e^{1 + i\pi/3} + e^{1 - i\pi/3} = e.\)

19. If \((\frac{x}{a})^n + (\frac{y}{b})^n = 2,\) then find \(\frac{dy}{dx}\) at \((a, b)\).

20. Find the length of the sub-tangent to the curve \(x^3 + xy + y^2 = 19\) at \((1, 3)\).
21. Evaluate \( \int \frac{1}{\sin^2 x \cos^2 x} \, dx \).

22. Form the differential equation by eliminating the parameter \( c \).

\[ \sin^{-1} x + \sin^{-1} u = c. \]

\[ \text{PART – C} \]

I. Answer any three questions: \( 3 \times 5 = 15 \)

23. Find the number of all positive divisors and the sum of all positive divisors of 39744.

24. a) Show that

\[
\begin{vmatrix}
 a^2 + bc & a & 1 \\
 b^2 + ca & b & 1 \\
 c^2 + ab & c & 1 \\
\end{vmatrix} = -2 (a-b)(b-c)(c-a).
\]

b) Find the values of \( x \) and \( y \) according to Cramer's rule:

\[ x + 2y = 7 \]
\[ 4x - 5y = 2. \]

25. a) Prove that the set \( H = \{ 1, 2, 4 \} \otimes_7 \) is a sub-group of the group \( G = \{ 1, 2, 3, 4, 5, 6 \} \otimes_7 \) under multiplication modulo 7.

b) Prove that the identity element of a group is unique.

26. a) If the vectors \( \hat{i} - \hat{j} + \lambda \hat{k}, \ 4\hat{i} + 2\hat{j} + 9\hat{k}, \ 5\hat{i} + \hat{j} + 14\hat{k} \)

and \( 3\hat{i} + 2\hat{j} + 7\hat{k} \) are the position vectors of the four coplanar points, find \( \lambda \).

b) Find the unit vector in the direction of \( 2\hat{i} - \hat{j} + 2\hat{k} \).
II. Answer any two questions: \(2 \times 5 = 10\)

27. a) Find the equation of the circle which cuts the two circles 
\[ x^2 + y^2 - 6y + 1 = 0 \text{ and } x^2 + y^2 - 4y + 1 = 0 \]
and whose centre lies on the line \(3x + 4y + 5 = 0\). \(3\)

b) Find the equation of the circle having \((4, 2)\) and \((-5, 7)\) as end points of the diameter. \(2\)

28. a) Find the condition for the line \(y = mx + c\) to be a tangent to the hyperbola 
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \] \(3\)

b) Find the focus of the parabola \(y^2 - 8x - 32 = 0\). \(2\)

29. Prove that
\[ \tan^{-1} \left( \sqrt{\frac{a(a+b+c)}{bc}} \right) + \tan^{-1} \left( \sqrt{\frac{b(a+b+c)}{ca}} \right) + \tan^{-1} \left( \sqrt{\frac{c(a+b+c)}{ab}} \right) = \pi \] \(5\)

III. Answer any three of the following questions: \(3 \times 5 = 15\)

30. a) Differentiate \(\cosec(ax)\) w.r.t. \(x\) from the first principle. \(3\)

b) Differentiate \(\sin x\) with respect to \(\log x\). \(2\)

31. a) If \(e^x + e^y = e^{x-y}\) prove that \(\frac{dy}{dx} = -e^{y-x}\). \(2\)

b) If \(x = \tan^{-1} \sqrt{\frac{1-t}{1+t}}, \ y = \cos^{-1} \left( 4t^3 - 3t \right)\), prove that \(\frac{dy}{dv} = 6\). \(3\)
32. a) If \( y = \sin^2 \left( \cot^{-1} \sqrt{\frac{1 + x}{1 - x}} \right) \), prove that \( \frac{dy}{dx} = -\frac{1}{2} \).  

\[ \text{b) Evaluate } \int \frac{\sin x}{1 + \sin x} \, dx. \]  

33. a) Evaluate \( \int \frac{\cos x}{2 \sin^2 x + 3 \sin x + 4} \, dx. \)  

\[ \text{b) Evaluate } \int \frac{x}{\sqrt{x^2 - 4}} \, dx. \]  

34. Find the area of the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \) by integration method.  

**PART - D**

Answer any **two** of the following questions:  

2 \times 10 = 20

35. a) Define an ellipse. Derive the equation of the ellipse in the standard form

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]  

\[ \text{b) If } A = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}, \text{ find } A^{-1} \text{ by Cayley-Hamilton theorem.} \]  

36. a) State and prove D’Moivre’s theorem for rational index.  

\[ \text{b) Prove that the sine rule } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ by vector method.} \]  

37. a) Prove that the greatest size rectangle that can be inscribed in a circle of radius \( a \) is a square.  

\[ \text{b) Find the general solution of} \]
( \sqrt{3} + 1 ) \cos \theta + ( \sqrt{3} - 1 ) \sin \theta = 2. 

38. a) Prove that
\[ \int_{0}^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log \left( \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right). \]

b) Solve the differential equation
\[ \frac{dy}{dx} = (x + y - 1)^2. \]

PART - E

Answer any one of the following questions: 1 × 10 = 10

39. a) Find the cube roots of 1 + i and represent the Argand diagram.

b) Find the length of the chord intercepted by the circle
\[ x^2 + y^2 - 8x - 6y = 0 \text{ and the line } x - 7y - 8 = 0. \]

c) Find the digit in the unit place of 7^{123}.

40. a) If \( |\vec{a}| = 13, |\vec{b}| = 19, |\vec{a} + \vec{b}| = 24\), find \( |\vec{a} - \vec{b}| \).

b) Find \( \int \tan^4 x \, dx \).

c) If \( y = \log \sqrt{\cos x} \), find \( \frac{dy}{dx} \).