Second PUC    July – 2007 Question paper

MATHEMATICS

PART - A

Answer all the ten questions :                     10 x 1 = 10

1. If \( 3^{127} = x \pmod{10} \), find \( x \).

2. If \( A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \), find \( AB \).

3. In a group \(( G, \ast )\), if \( a \ast x = e \forall a \in G \), find \( x \).

4. Find the value of \(( j - 3k ) \times ( i - j + 2k )\).

5. Find the centre of the circle passing through \((0, 0)\), \((3, 0)\) and \((0, 5)\).

6. Find the vertex of parabola \(( y - 2)^2 = -8x\).

7. If \( \cos^{-1} x = \sin^{-1} x = 0 \), prove that \( x = \frac{1}{\sqrt{2}} \).

8. Find amplitude of \( 2\ell - 4 \).

9. If \( y = 3^{-x} \), find \( \frac{dy}{dx} \).

10. Evaluate: \( \int_0^{\pi/2} \sqrt{1 - \cos 2x} \, dx \).

PART - B

Answer any ten questions :                     10 x 2 = 20
11. If \( a \equiv b \pmod{m} \) and \( n \mid m \ \forall \ n \in I \), prove that \( a \equiv b \pmod{n} \).

12. Without expansion, find the value of

\[
\begin{vmatrix}
\sin^2 x & \cos^2 x & 1 \\
\cos^2 x & \sin^2 x & 1 \\
-10 & 12 & 2
\end{vmatrix}
\]

13. If \( \mathcal{Q}^+ \) is the set of all positive rationals w.r.t. \(*\),

define \( a * b = \frac{2ab}{3} \ \forall \ a, b \in \mathcal{Q}^+ \). Find

a) Identity element.

b) Inverse of \( a \) under \(*\).

14. For any vector \( \vec{a} \), prove that

\[
\vec{a} = (\vec{a} \cdot i)i + (\vec{a} \cdot j)j + (\vec{a} \cdot k)k.
\]

15. Find the length of tangent from the centre of circle \( x^2 + y^2 - 8x = 0 \) to the circle \( 3x^2 + 3y^2 = 7 \).

16. Find the centre of ellipse whose vertices are \( (2, -2) \) and \( (2, 4) \). Also find the length of major axes.

17. If \( \tan^{-1} x + \tan^{-1} y = \frac{\pi}{2} \), prove that \( xy = 1 \).

18. If \( x = \text{cis} \alpha \) and \( y = \text{cis} \beta \),
prove that $\sin (\alpha - \beta) = \frac{1}{2i} \left( \frac{x}{y} - \frac{y}{x} \right)$.

19. If $y \log_e x = y - x$, prove that

$$\frac{dy}{dx} = \frac{2 - \log_e x}{\left( 1 - \log_e x \right)^2}.$$

20. Prove that $x^x$ is minimum at $x = \frac{1}{e}$.

21. Evaluate:

$$\int \frac{1}{5e^{3x} + 1} \, dx.$$

22. Form a differential equation for the equation $x^2 + y^2 + 2ky = 0$.

**PART - C**

1. Answer any three questions: $3 \times 5 = 15$

23. a) Find the G.C.D. of 48 and 18. If $6 = 48m + 18n$, find $m$ and $n$. $3$

b) Solve $51x = 32 \mod 7$. Write the solution set. $2$

24. If

$$\begin{bmatrix} 7 & 6 & -5 \\ 3 & -4 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ 10 \end{bmatrix},$$

find $x$, $y$ and $z$ using Cramer's Rule. $5$
25. Prove that the set \( G = \{ \ldots, 5^{-2}, 5^{-1}, 5^{0}, 5^{1}, 5^{2}, \ldots \} \) is an 
Abelian group under usual multiplication.

26. a) Find the area of the triangle \( ABC \) where position vectors of \( A, B, C \) are \( i - j + 2k, \ 2j + k, \ j + 3k \) respectively.

b) Prove that

\[
\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}.
\]

II. Answer any two questions: \( 2 \times 5 = 10 \)

27. a) Obtain the condition for two circles

\[
x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0
\]

\[
x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0
\]

to intersect orthogonally.

b) The radical axis of two circles is \( x - 2y + 6 = 0 \). The equation of one of the circles is \( 2x^2 + 2y^2 - 8x - 4y - 22 = 0 \). If the second circle passes through the point \((1, 6)\), find its equation.

28. a) Find the centre and the foci of ellipse

\[
4x^2 + 9y^2 + 16x - 18y - 11 = 0.
\]
b) Find the focal distance of any point \((x, y)\) on the parabola 
\[y^2 = 4ax.\]

29. a) Prove that 
\[
\tan \left( \frac{1}{2} \sin^{-1} \left( \frac{2x}{1 + x^2} \right) + \frac{1}{2} \cos^{-1} \left( \frac{1 - x^2}{1 + x^2} \right) \right) = \frac{2x}{1 - x^2}.
\]

b) Find the general solution of 
\[\tan m\theta = \tan n\theta.\]

III. Answer any three of the following questions: 

30. a) Differentiate \(\cosec 4x\) with respect to \(x\) from first principles. 

b) If \(y = \tan^{-1} \left( \frac{2 + 5 \tan x}{5 - 2 \tan x} \right)\), find \(\frac{dy}{dx}\). 

31. a) If \(y = \left[ x + \sqrt{1 + x^2} \right]^m\), prove that 
\[
\left(1 + x^2\right) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - m^2 y = 0.
\]

b) Find a point on the curve \(y = x^3 - 3x\), where tangent is parallel to the line joining the points \((1, -2)\) and \((2, -5)\).
32. a) A circular blot of ink in a blotting paper increases in area in such a way that the radius \( r \) cm at time \( t \) seconds is given by
\[ r = 2t^2 - \frac{t^3}{4}. \]
Find the rate of increase of area when \( t = 2 \).

b) Prove that \[ \int uv' \, dx = uv - \int vu' \, dx \]
where \( u' = \frac{du}{dx} \) and \( v' = \frac{dv}{dx} \).

33. a) Evaluate:
\[ \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx \]

b) Evaluate:
\[ \int \frac{1}{\sqrt{1 - 4x - 4x^2}} \, dx \]

34. Find the area enclosed between the parabolas \( y^2 = 4ax \) and \( x^2 = 4ay \).

PART - D

Answer any two of the following questions:

35. a) Define director circle of a hyperbola. Derive the equation of director circle of the hyperbola.

b) Using \( A(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \)
find \( \text{adj} \left[ A(x) \right] \). Prove that \( \text{adj} \left[ A(x) \right] = A(-x) \).

36. a) Find the fourth roots of \( (\sqrt{3} - i)^3 \). Also find their continued product.

b) Prove by vector method,
\[
\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta.
\]

37. a) Show that the height of a right circular cylinder of the greatest volume which is inscribed in a sphere of radius \( a \) is \( \frac{2a}{\sqrt{3}} \). Find the radius of the right circular cylinder.

b) Find the general solution of
\[
\sec x - \tan x + \sqrt{3} = 0
\]

38. a) Prove that
\[
\int_{0}^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}
\]

b) Solve the differential equation
\[
\frac{dy}{dx} = \tan^2 (x + y)
\]